EDUCATION Revista Mexicana de Física E **64** (2018) 26–38 JANUARY–JUNE 2018

The retarded potential of a non-homogeneous wave equation: introductory analysis through the Green functions

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Received 17 August 2017; accepted 11 September 2017

The retarded potential, a solution of the non-homogeneous wave equation, is a subject of particular interest in many physics and engineering applications. Examples of such applications may be the problem of solving the wave equation involved in the emission and reception of a signal in a synthetic aperture radar (SAR), scattering and backscattering, and general electrodynamics for media free of magnetic charges. However, the construction of this potential solution is based on the theory of distributions, a topic that requires special care and time to be understood with mathematical rigor. Thus, the goal of this study is to provide an introductory analysis, with a medium level of formalism, on the construction of this potential solution and the handling of Green functions represented by sequences of well-behaved approximating functions.

Keywords: Mathematical methods in physics; mathematics; diffraction theory; backscattering; radar.

PACS: 01.30.Rr; 02.30.Em; 41.20.Jb

1. Introduction

Potential theory can be simply understood as the art of solving a linear distributional non-homogeneous differential equation through the Green functions [1–3]. In the context of this study, our interest resides in the construction of an integral solution that derives from the divergence Gauss theorem, certain Green identities, and the handling of the Green functions. In the construction of this potential solution we also find other results of great importance, such as the integral theorem of Helmholtz and Kirchhoff, which is the main result that supports the scalar diffraction theory in optics [4]. However, Green functions are not properly functions in the usual sense, since they are formally defined as *distributions*. Distributional theory, Green functions, and the use of Green identities have been successfully implemented in many theoretical and applied works, *e.g.*, SAR theory [5–7], scattering and wave propagation [8–12], wave diffraction and electrodynamics [13–16], phase unwrapping [17–20], etc.

The concept of distribution is not easy to explain and in most of the references where distributions are mentioned, such as basic courses in calculus [21], differential equations [22], linear systems [23], or Fourier analysis [24], a detailed explanation of such abstractions is not usually given. Hence, in order to have a more solid notion of the concept of distributions, or *generalized functions*, specialized literature should be consulted. This literature is specifically related to a coarse field of mathematics called functional analysis [25, 26] and some notions on Lebesgue measure [27]. Certainly, in this work we will not provide specific and detailed information about distributions. Instead of that, we are going to use the artifice of working with sequences of well-behaved approximating functions [28], which permit us to talk about distributions concisely and without too much complexity. However, we hope to motivate the reader in the study of distributions through one of their most important applications: the standard solution of the non-homogeneous wave equation, also known as the retarded potential. To understand the construction of this integral solution, we first need to expose the main problem involved with the non-homogeneous wave equation. Thus, in order to establish the context of such a problem, we start in Sec. 2 with a typical deduction of the non-homogeneous wave equation from the Maxwell equations. Motivated in the perspective of SAR theory, in the subsequent sections, we provide some descriptive guidelines for constructing the potential solution of this non-homogeneous wave equation. The potential solution is supported by the divergence Gauss theorem and the Green identities, described in Sec. 3. Of course, the definition of Green functions is also explained in Sec. 4, where we discuss about certain incongruities when working with Green functions, found in certain references (for example [21]). These incongruities refer to the question: How can be demonstrated that a function is a Green function with some of formalism? We emphasize that we do not want to criticize the descriptive style of references [21–23] or [24], which are indeed quite advisable. We simply want to point out that the treatment of distributions should be made with more care. Additionally to this discussion, a description of the integral theorem of Helmholtz and Kirchhoff, and its relation with the retarded potential is presented in Sec. 5. Finally, Sec. 6 outlines our conclusions.

2. Derivation of the non-homogeneous wave equation

It is well known that, an electromagnetic wave, such as a radar wave, is characterized in each point of the space $\hat{x} = (x, y, z)$, and each time t, by the vectorial functions $E = E(t, \hat{x})$, the *electric field* (EF), and $H = H(t, \hat{x})$, the *magnetic field* (MF). When assuming as known the variables $J = J(t, \hat{x})$, the *current density* (vectorial field), $\sigma = \sigma(t, \hat{x})$, the *charge density* (scalar function), q, the *permittivity* constant, and p, the *permeability* constant, the fields E and H can be found. These fields are determined by the *Maxwell equations*: $\nabla \cdot E = \sigma/q$, the Gauss law for EF, $\nabla \cdot H = 0$, the Gauss law for MF, $\nabla \times E + p(\partial H/\partial t) = \hat{0}$, the Faraday law, and $\nabla \times H - q(\partial E/\partial t) = J$, the Ampère law. Here, $\nabla := (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, $\hat{0} = (0, 0, 0)$, and the media is assumed free of magnetics sources. In this case, operations $\nabla \cdot F$ and $\nabla \times F$ refer, respectively, to the *divergence* and the *rotational* of any vectorial field $F = F(t, \hat{x})$. Hence, a method to find E and H can be constructed from the previous Maxwell equations and the next theorems:

Theorem 1: Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a real valued vectorial field of class C^1 (continuous function with continuous first order partial derivatives), exceptionally in a finite number of points. Then, F is the gradient of some scalar function f (that is, $F = \nabla f$), if and only if $\nabla \times F = \hat{0}$.

Theorem 2: If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vectorial field such that $\nabla \cdot F = 0$, then, there is a C^1-vectorial field G such that $F = \nabla \times G$.

These theorems are demonstrated in [21] for real valued functions dependent on variable $\hat{x}$ (where $F = F(\hat{x})$), however, they can be generalized to complex valued functions of the form $F : \mathbb{R}^3 \to \mathbb{C}^3$. Moreover, these theorems are independent of variable t, so, they are also true for complex fields [29] of the form $F : \mathbb{R}^4 \to \mathbb{C}^3$ where $F = F(t, \hat{x})$. Now, by assuming E and H as C^2-fields (that means, C^1-fields with continuous second order partial derivatives), from the Gauss law for MF, and Theorem 2, there exists a field A_0 such that

$$H = \nabla \times A_0. \tag{1}$$

Additionally, from Faraday law and Eq. (1), we have that

$$\begin{aligned}\hat{0} &= \nabla \times E + p\frac{\partial}{\partial t}(\nabla \times A_0)\\ &\vdots\\ &= \nabla \times \left(E + p\frac{\partial A_0}{\partial t}\right),\end{aligned} \tag{2}$$

which, from Theorem 1, implies the existence of a function μ_0 satisfying

$$E + p\frac{\partial A_0}{\partial t} = \nabla \mu_0. \tag{3}$$

On the other hand, from the Ampère law, Eqs.(1) and (3), and the vectorial identity $\nabla \times (\nabla \times A_0) = \nabla(\nabla \cdot A_0) - \nabla^2 A_0$ (where $\nabla^2 := (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$ is the Laplacian operator), it follows that

$$\begin{aligned}J &= \nabla \times H - q\frac{\partial E}{\partial t}\\ &= \nabla \times (\nabla \times A_0) - q\frac{\partial}{\partial t}\left(\nabla \mu_0 - p\frac{\partial A_0}{\partial t}\right)\\ &= \nabla(\nabla \cdot A_0) - \nabla^2 A_0 - q\frac{\partial}{\partial t}(\nabla \mu_0) + pq\frac{\partial^2 A_0}{\partial t^2}.\end{aligned} \tag{4}$$

From the previous relation, it is obtained that

$$\begin{aligned}\nabla^2 A_0 - pq\frac{\partial^2 A_0}{\partial t^2} &= -J + \nabla(\nabla \cdot A_0) - q\frac{\partial}{\partial t}(\nabla \mu_0)\\ &= -J + \nabla(\nabla \cdot A_0) - \nabla\left(q\frac{\partial \mu_0}{\partial t}\right)\\ &= -J + \nabla\left(\nabla \cdot A_0 - q\frac{\partial \mu_0}{\partial t}\right).\end{aligned} \tag{5}$$

Now, by using the Gauss law for EF and Eq. (3), we get

$$\nabla^2 \mu_0 = \frac{\sigma}{q} + p\frac{\partial}{\partial t}(\nabla \cdot A_0). \tag{6}$$

Let us define f as a scalar solution of the equation

$$\nabla^2 f - pq\frac{\partial^2 f}{\partial t^2} = -\left(\nabla \cdot A_0 - q\frac{\partial \mu_0}{\partial t}\right), \tag{7}$$

and μ, as the function $\mu := p(\partial f/\partial t) + \mu_0$. From these definitions we obtain the relation

$$\frac{\partial f}{\partial t} = \frac{(\mu - \mu_0)}{p}. \tag{8}$$

Thus, it can be noticed that

$$H = \nabla \times A, \tag{9}$$

with $A = A_0 + \nabla f$. The result in Eq. (9) is logical because $\nabla \times (\nabla f) = \hat{0}$ for all differentiable scalar field f; in other words, $\nabla \times A = \nabla \times A_0$ for such A. Since H can be calculated, as suggested in Eq. (9), there is a function μ_1 that satisfies

$$E + p\frac{\partial A}{\partial t} = \nabla \mu_1, \tag{10}$$

in analogy to the steps given for deriving Eq. (3). However, $\nabla \mu = \nabla \mu_1$ due to the fact that

$$\begin{aligned}\nabla \mu_1 &= E + p\frac{\partial}{\partial t}(A_0 + \nabla f)\\ &= \left(E + p\frac{\partial A_0}{\partial t}\right) + p\frac{\partial}{\partial t}(\nabla f)\\ &= \nabla \mu_0 + p\nabla\left(\frac{\partial f}{\partial t}\right) = \nabla \mu,\end{aligned} \tag{11}$$

a consequence of Eqs. (3), (8), and (10). In this case, we can replace $\nabla \mu_1$ by $\nabla \mu$ in Eq. (10) in order to obtain

$$E + p\frac{\partial A}{\partial t} = \nabla \mu. \tag{12}$$

Thus, by using Eqs. (9) and (12), this A must satisfy

$$\nabla^2 A - pq\frac{\partial^2 A}{\partial t^2} = -J + \nabla\left(\nabla\cdot A - q\frac{\partial \mu}{\partial t}\right) \tag{13}$$

and

$$\nabla^2 \mu = \frac{\sigma}{q} + p\frac{\partial}{\partial t}(\nabla\cdot A), \tag{14}$$

in analogy to the steps for concluding Eqs. (5) and (6) from Eqs. (1) and (3). However, we have now a simplification in our calculations because

$$\nabla\cdot A - q\frac{\partial \mu}{\partial t} = 0, \tag{15}$$

as the reader can confirm.

Since Eq. (15) takes place, then Eq. (13) reduces to

$$\nabla^2 A - pq\frac{\partial^2 A}{\partial t^2} = -J. \tag{16}$$

In a similar fashion, from Eq. (15), the expression in Eq. (14) is rewritten as

$$\nabla^2 \mu - pq\frac{\partial^2 \mu}{\partial t^2} = \frac{\sigma}{q}. \tag{17}$$

Therefore, by considering the *wave propagation velocity* $c_0 := 1/\sqrt{pq}$ (as defined in [30]), and introducing the function $\zeta := -\sigma/q$, Eqs. (16) and (17) are correspondingly expressed as

$$\frac{1}{c_0^2}\frac{\partial^2 A}{\partial t^2} - \nabla^2 A = J, \tag{18a}$$

$$\frac{1}{c_0^2}\frac{\partial^2 \mu}{\partial t^2} - \nabla^2 \mu = \zeta. \tag{18b}$$

Then, when assuming J and ζ as known functions, the solutions of Eqs. (18) for A and μ, permit us to find H and E, from Eqs. (9) and (12), respectively. Equation (18) for μ, and analogously for A, is known as the *non-homogeneous wave equation*, or *d'Alembert equation* [5] in the *distributional sense*. Due to its undulatory nature, the solution of this equation is called *scalar wave* in the case of μ, or *vectorial wave*, in the case of A.

3. The Helmholtz equation, the divergence Gauss theorem, and the Green identities

For constructing a solution of the non-homogeneous wave equation, from the methods applied for solving differential equations [22, 31, 32], it is typically proposed a general solution of the form $\mu = \mu_0 + \mu_1$, where μ_0 is solution of the homogeneous version of the wave equation (when $\zeta = 0$), and μ_1 is solution of the original non-homogeneous equation (when $\zeta \neq 0$). In radar language [33–35], function ζ is interpreted as the source or the electric pulse emitted by an antenna. In this context, when $\zeta \neq 0$, it is understood that the pulse induces a propagating wave which reaches an object. This wave is known as *emitting* or *incident field* and it is denoted by μ_1. In opposite form, the object, assumed to be like a non-emitting electric pulse source ($\zeta = 0$), reflects or backscatters the incident wave. So, the reflected wave or *backscattered field* is denoted by μ_0 (details of this conception are explained in [5]). Consequently, the general solution μ is known as *total field*. Since the total field can be found by solving first the homogeneous wave equation, we are going to focus our attention in the construction of μ_0. Thus, in this section, variable μ_0 will be denote as μ for simplifying notation. Moreover, it is important to remark that any component of the vectorial function A in Eq.(18), with $J = \hat{0}$, can be found just as solving the scalar case for μ in Eq.(18), with $\zeta = 0$. In this sense, it will be sufficient to establish the theory for solving this scalar case.

Let us consider a scalar wave with spatial period or wavelength λ_0, refractive index media $n = 1$ (air) and angular-temporal frequency ω_0. Such wave can be represented by the function $\tilde{\mu}(t,\hat{x}) = a(\hat{x})\cos(\omega_0 t + \phi(\hat{x}))$, where $a(\hat{x})$ is the amplitude, and $\phi(\hat{x})$ is the phase. This cosinusoidal form is well known from the classic theory for solving the homogeneous wave equation; specifically speaking, the method called *separation of variables*, and the *Fourier series* [22]. However, in an equivalent way, the form of the wave can be generalized to the complex representation $\mu(t,\hat{x}) = f(\hat{x})e^{i\omega_0 t}$, where $\tilde{\mu}$ is the real part of μ, $f(\hat{x}) = a(\hat{x})e^{i\phi(\hat{x})}$ is the spatial part of μ (the so-called *phasor* or *complex perturbation* [4]), and i is the imaginary unit. Since this complex representation must satisfy the homogeneous wave equation, we have that $\nabla^2\mu - (1/c_0^2)(\partial^2\mu/\partial t^2) = 0$, where $c_0 = \lambda_0/T_0$ is the wave propagation velocity and T_0 denotes the temporal period of such wave. In this case $\tau_0 = 1/T_0$ would be called simply as the temporal frequency, where $\omega_0 = 2\pi\tau_0$. Thus, when substituting the complex representation of μ in the homogeneous wave equation, we obtain that $e^{i\omega_0 t}\nabla^2 f(\hat{x}) - (i^2\omega_0^2/c_0^2)f(\hat{x})e^{i\omega_0 t} = 0$, which implies

$$(\nabla^2 + k_0^2)f(\hat{x}) = 0. \tag{19}$$

This last expression is called *Helmholtz equation*, where $k_0 = \omega_0/c_0 = (2\pi\tau_0)/(\lambda_0\tau_0) = (2\pi)/\lambda_0$ is the wave number or the angular-spatial frequency.

On the other hand, a well-known result in vectorial calculus is the *divergence Gauss theorem* [21], which can be written as

$$\int_{\Omega}^{3} \nabla\cdot F dV = \int_{\partial\Omega}^{2} F\cdot\hat{n} dA, \tag{20}$$

where $F = F(\hat{x})$ is a C^1-vectorial field on Ω such that $F : \mathbb{R}^3 \to \mathbb{R}^3$, and Ω is an elemental region in $\mathbb{R}^3$ with positive parametrized boundary $\partial\Omega$. The surface $\partial\Omega$ can be a sphere, an ellipsoid, a parallelepiped, etc. In this theorem, symbol $\int_{\Omega}^{3}$ denotes the triple integral over the region Ω, $\int_{\partial\Omega}^{2}$ denotes the double integral over the surface $\partial\Omega$, dV is a volume differential element, dA is an area differential element, and symbol $\cdot$ denotes dot product. In addition, vectorial function $\hat{n}$ represents the unitary normal vector with respect to the surface $\partial\Omega$, in such a way that, it points towards the outside of the surface. So, if we take $F = f\nabla g$, where $f = f(\hat{x})$ and $g = g(\hat{x})$ are two differentiable scalar fields from $\mathbb{R}^3$ to $\mathbb{R}$, then

$$\nabla \cdot F = \nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g. \qquad (21)$$

When substituting this equation in the divergence Gauss relation, and introducing the directional derivative $(\partial g/\partial n) := \nabla g \cdot \hat{n}$, we obtain

$$\overset{3}{\int\limits_{\Omega}} \nabla f \cdot \nabla g dV + \overset{3}{\int\limits_{\Omega}} f\nabla^2 g dV = \overset{2}{\int\limits_{\partial\Omega}} f\frac{\partial g}{\partial n} dA. \qquad (22)$$

Equation (22) is known as the *first Green identity*. Analogously, when considering $F = g\nabla f$, the equality

$$\overset{3}{\int\limits_{\Omega}} \nabla g \cdot \nabla f dV + \overset{3}{\int\limits_{\Omega}} g\nabla^2 f dV = \overset{2}{\int\limits_{\partial\Omega}} g\frac{\partial f}{\partial n} dA, \qquad (23)$$

is deduced. By subtracting Eq. (23) from Eq. (22), it is concluded that

$$\overset{3}{\int\limits_{\Omega}} (f\nabla^2 g - g\nabla^2 f) dV = \overset{2}{\int\limits_{\partial\Omega}} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right) dA, \qquad (24)$$

a relation called *second Green identity*. It is important to remark that the divergence Gauss theorem is also valid for C^1-complex vectorial fields $F : \mathbb{R}^3 \to \mathbb{C}^3$ that can be expressed in terms of complex scalar fields $f, g : \mathbb{R}^3 \to \mathbb{C}$. This last includes the possibility of considering fields of the form $F = F(t, \hat{x})$, $f = f(t, \hat{x})$, and $g = g(t, \hat{x})$, which means that $F : \mathbb{R}^4 \to \mathbb{C}^3$, and $f, g : \mathbb{R}^4 \to \mathbb{C}$. The validity of this theorem is a consequence of the linearity of the integrals for complex valued expressions that can be denoted as $F = \mathrm{Re}(F) + i\mathrm{Im}(F)$, where $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$, are the real and the complex parts of F, respectively. Thus, the only required condition is to have $\mathrm{Re}(F)$ and $\mathrm{Im}(F)$, as C^1-real valued functions.

4. The Green functions: language of distributions

Informally, any function g that satisfies the Helmholtz equation *almost everywhere* [26,27] on $\mathbb{R}^3$, could be called *Green function* [4], however, such g requires to satisfy another properties in the context of *distributions* [26]. In this sense, expression *almost everywhere* refers to a property which is satisfied at all points of a domain with the exception of the points in a zero volume subset of the domain. For this particular case, the property would be the fulfilling of the Helmholtz equation. On the other hand, the Green functions and the Green identities are important and useful, specially in optics, for establishing a transcendental theorem: the *integral theorem of Helmholtz and Kirchhoff*. This theorem is the key result for supporting the *scalar diffraction theory* and it is related in part with the solution of a particular distributional non-homogeneous wave equation, the so-called *retarded potential* of the *d'Alembert equation* [5]. Now, let L be a linear operator applied to scalar functions depending on $\hat{x}$. Formally, a distribution $G(\hat{x}, \hat{y})$ is said to be a *Green function* with respect to L, if it satisfies: a) $G(\hat{x}, \hat{y}) = G(\hat{y}, \hat{x})$ for all $\hat{x} = (x, y, z)$ and $\hat{y} = (u, v, w)$ in $\mathbb{R}^3$, and b) $L[G(\hat{x}, \hat{y})] = \delta(\hat{y} - \hat{x})$, where $\delta(\hat{y} - \hat{x})$ is the *Dirac delta distribution*. However, every distribution D is said to be a Dirac delta, if it satisfies: 1) $D(\hat{y} - \hat{x}) = 0$, for all $\hat{y} \neq \hat{x}$, and 2)

$$\overset{3}{\int\limits_{\mathbb{R}^3}} D(\hat{y} - \hat{x}) dV(\hat{y}) = 1.$$

In this definition $dV(\hat{y})$ refers to a volume differential element with respect to the integration variable $\hat{y}$. Thus, when an arbitrary distribution D fulfills properties 1) and 2), we write $\delta := D$.

Let us consider $G(\hat{x}, \hat{y}) := g_{\hat{y}}(\hat{x})$, where $g_{\hat{y}}(\hat{x}) = e^{ik_0||\hat{x}-\hat{y}||}/||\hat{x} - \hat{y}||$. Here, $||\cdot||$ denotes the Euclidean norm for vectors in $\mathbb{R}^3$. This function satisfies $g_{\hat{y}}(\hat{x}) = g_{\hat{x}}(\hat{y})$ by symmetry, then $G(\hat{x}, \hat{y})$ fulfills property a). Is this G a Green function? What linear operator could be related to this G to declare it as a Green function? Well, the obvious answer is that such operator must be involved with the Helmholtz equation. So, if we consider $L := [-1/(4\pi)](\nabla^2 + k_0^2)$, then $L[f(\hat{x})] = [-1/(4\pi)](\nabla^2 + k_0^2)f(\hat{x})$ for any smooth function $f : \mathbb{R}^3 \to \mathbb{C}$. Moreover, $D(\hat{y} - \hat{x}) := L[G(\hat{x}, \hat{y})] = [-1/(4\pi)](\nabla^2 + k_0^2)g_{\hat{y}}(\hat{x}) = 0$ for all $\hat{x} \neq \hat{y}$, from Proposition 1 in Appendix A. This also implies that property 1) is satisfied by $D(\hat{y} - \hat{x})$, every time that $\hat{x} \neq \hat{y}$. Thus, assuming this D as a simple function depending on $\hat{y}$, which is a discontinuous function in $\hat{y} = \hat{x}$, we rigorously have that

$$\overset{3}{\int\limits_{\mathbb{R}^3}} D(\hat{y} - \hat{x}) dV(\hat{y}) = 0$$

by using improper integrals. Even in the case of more complex integrals, if a function is zero almost everywhere in certain domain, then its Lebesgue integral on such domain should be zero [27]. Thus, since property 2) is not achieved by this D, it implies that $G(\hat{x}, \hat{y}) = g_{\hat{y}}(\hat{x})$ is not a Green function. But, why so many references [2–4,8] declare $g_{\hat{y}}(\hat{x})$ as a Green function? Well, may be the answer is in the interpretation of D, and consequently G, as distributions. In the spirit of considering a sequence of well-behaved approximating functions (see the Remarks in Appendix A), G can be rewritten to the equivalent form

$$G(\hat{x}, \hat{y}) := \lim_{\gamma \to 0^+} e^{ik_0 r} f_\gamma(r), \qquad (25)$$

where $r = ||\hat{x} - \hat{y}||$ and $\lim\limits_{\gamma \to 0^+} f_\gamma(r) = 1/r$ for $r > 0$. In this case for each $\gamma > 0$, f_γ is a smooth function for all $r > 0$ and right continuous at $r = 0$. In other words, df_γ/dr exists for all $r > 0$, and $\lim_{r\to 0^+} f_\gamma(r) = f_\gamma(0)$, respectively. Now, it is evident that G in Eq. (25) satisfies properties a) and 1), however, if property 2) is required for D with this new G, we need to construct function f_γ conveniently. There are many forms to do this construction, but we are going to propose a particular one. Before starting with this construction, let us think again that $G(\hat{x},\hat{y})$ is $g_{\hat{y}}(\hat{x})$, and consider the next argumentation:

From property 1), it is inferred that the integral of D on $\mathbb{R}^3$ is the same that the integral of D on any open ball $B_\varepsilon(\hat{x}) = \{\hat{y}\in\mathbb{R}^3 : ||\hat{y}-\hat{x}|| < \varepsilon\}$. Then,

$$\int\limits_{\mathbb{R}^3}^{3} D(\hat{y}-\hat{x})dV(\hat{y}) = \left(\frac{-1}{4\pi}\right) \times \int\limits_{B_\varepsilon(\hat{x})}^{3} (\nabla^2 + k_0^2) g_{\hat{y}}(\hat{x})dV(\hat{y}) = \left(\frac{-1}{4\pi}\right) \times \left[\int\limits_{B_\varepsilon(\hat{x})}^{3} \nabla^2 g_{\hat{y}}(\hat{x})dV(\hat{y}) + k_0^2 \int\limits_{B_\varepsilon(\hat{x})}^{3} g_{\hat{y}}(\hat{x})dV(\hat{y})\right], \quad (26)$$

for all $\varepsilon > 0$ fixed. Thus, when considering a translation to the origin, a change to spherical coordinates, and the use of improper integrals, we have that

$$\int\limits_{B_\varepsilon(\hat{x})}^{3} g_{\hat{y}}(\hat{x})dV(\hat{y}) = \int\limits_{B_\varepsilon(\hat{x})}^{3} g_{\hat{x}}(\hat{y})dV(\hat{y}) = \frac{4\pi}{k_0^2}[e^{ik_0\varepsilon}(1-ik_0\varepsilon)-1], \quad (27)$$

independently of the discontinuity of $g_{\hat{x}}(\hat{y})$ at $\hat{y}=\hat{x}$. Now,

$$\int\limits_{B_\varepsilon(\hat{x})}^{3} \nabla^2 g_{\hat{y}}(\hat{x})dV(\hat{y}) = \int\limits_{B_\varepsilon(\hat{x})}^{3} \nabla\cdot\nabla g_{\hat{y}}(\hat{x})dV(\hat{y}) = -\int\limits_{B_\varepsilon(\hat{x})}^{3} \nabla_{\hat{y}}\cdot\nabla g_{\hat{y}}(\hat{x})dV(\hat{y}) = -\int\limits_{\partial B_\varepsilon(\hat{x})}^{2} \nabla g_{\hat{y}}(\hat{x})\cdot\hat{n}(\hat{y})dA(\hat{y}), \quad (28)$$

where $\hat{n}(\hat{y})$ is the unitary normal vector to the surface $\partial B_\varepsilon(\hat{x})$ and $dA(\hat{y})$ is an area differential element with respect to $\hat{y}$. This is a consequence of Proposition 2 in Appendix A and the divergence Gauss theorem in variable $\hat{y}$. Following the calculus of the previous triple integral (now a double integral) we have that

$$\ldots = -\int\limits_{\partial B_\varepsilon(\hat{x})}^{2} e^{ik_0 r}\left[\frac{ik_0 r - 1}{r^3}\right](\hat{x}-\hat{y}) \cdot \frac{(\hat{y}-\hat{x})}{||\hat{y}-\hat{x}||}dA(\hat{y}) = \int\limits_{\partial B_\varepsilon(\hat{x})}^{2} e^{ik_0 r}\left[\frac{ik_0 r-1}{r^3}\right] \times \left(\frac{r^2}{r}\right)dA(\hat{y}) = 4\pi e^{ik_0\varepsilon}(ik_0\varepsilon - 1), \quad (29)$$

where the values $r = ||\hat{x}-\hat{y}||$ in the double integrals are all equal to ε, because $\hat{y}\in\partial B_\varepsilon(\hat{x}) = \{\hat{y}\in\mathbb{R}^3 : ||\hat{y}-\hat{x}|| = \varepsilon\}$. When considering Eqs. (27), (28), and (29) in connection with Eq. (26), we get property 2). Therefore, function $g_{\hat{y}}(\hat{x})$ results to be a Green function.

The last argumentation, although desirable, is false and its main fail is: function $\nabla g_{\hat{y}}(\hat{x}) = e^{ik_0 r}(ik_0 r - 1)(\hat{x}-\hat{y})/r^3$ is not a C^1-function (a smooth function) on $B_\varepsilon(\hat{x})$, specifically at $\hat{y}=\hat{x}$ where $r=0$. Then, we have incorrectly applied the divergence Gauss theorem in Eq. (28), which requires smoothness for the vector field $\nabla g_{\hat{y}}(\hat{x})$. However, this mistake motivates us to think about an appropriate election of function f_γ in Eq. (25). The sequence of functions f_γ satisfy $\lim_{\gamma\to 0^+} e^{ik_0 r} f_\gamma(r) = e^{ik_0 r}/r = g_{\hat{x}}(\hat{y})$ for all $\hat{y}\neq\hat{x}$ $(r > 0)$, but that is not enough. For each $\gamma > 0$, we also need smoothness for the terms $\nabla e^{ik_0 r} f_\gamma(r)$ for all $\hat{y}\in B_\varepsilon(\hat{x})$. However, since the term $e^{ik_0 r}$ is smooth for all $r \geq 0$, then a sequence of functions of the form

$$f_\gamma(r) := \begin{cases} p_0(r) & r \geq \gamma \\ p_\gamma(r) & 0 \leq r < \gamma, \end{cases} \quad (30)$$

where $p_0(r) := 1/r$, could be useful. For each $\gamma > 0$, $p_\gamma(t)$ could be a convenient polynomial function, in such a way that $p_\gamma(\gamma) = p_0(\gamma)$, $p'_\gamma(\gamma) = p'_0(\gamma)$, and $p''_\gamma(\gamma) = p''_0(\gamma)$. Here, symbols $'$ and $''$ denote, correspondingly, the first and the second derivatives with respect to r. Hence, these conditions would warrant the smoothness of f_γ for all $r > 0$. However, if $p_\gamma(r)$ is a linear combination of r-powers and we want to analyze the behavior of $\nabla e^{ik_0 r} f_\gamma(r)$ (particularly at $r=0$), we first need to calculate the resultant expressions of the partial derivatives of $e^{ik_0 r} r^n$, with respect to x, y, and z. For instance, from the chain rule we have

$$\frac{\partial}{\partial x}[e^{ik_0 r} r^n] = e^{ik_0 r}[n r^{n-1} + ik_0 r^n]\frac{(x-u)}{r}, \quad (31)$$

for $\hat{x}\neq\hat{y}$ (or $r > 0$). This expression is not necessarily right continuous at $r=0$ when considering the definition of continuity by lateral limits. However, if we achieved to avoid divisions by r (removing the discontinuity at $r=0$), the formula in Eq. (31) would be better behaved. Thus, by considering integer powers of r such that $n \geq 2$ and defining the vector $\hat{i} := (1,0,0)$, we get

$$\frac{\partial}{\partial x}[e^{ik_0 r} r^n]\Big|_{\hat{x}=\hat{y}} := \lim_{h\to 0} \frac{e^{ik_0 r} r^n|_{\hat{x}=\hat{y}+h\hat{i}} - e^{ik_0 r} r^n|_{\hat{x}=\hat{y}}}{h}$$
$$= \lim_{h\to 0} \frac{e^{ik_0|h|}|h|^n}{h} = 0, \quad (32)$$

due to the fact that

$$\lim_{h\to 0^+} \frac{e^{ik_0|h|}|h|^n}{h} = \lim_{h\to 0^-} \frac{e^{ik_0|h|}|h|^n}{h} = 0,$$

as the reader can confirm. Moreover, from Eq.(31) it follows that

$$\lim_{\hat{x}\to\hat{y}} \frac{\partial}{\partial x}[e^{ik_0 r} r^n]$$
$$= \lim_{\substack{x\to u\\ r\to 0^+}} \left\{ e^{ik_0 r}[n r^{n-1} + ik_0 r^n] \frac{(x-u)}{r} \right\}$$
$$= \lim_{\substack{x\to u\\ r\to 0^+}} \left\{ e^{ik_0 r}[n r^{n-2} + ik_0 r^{n-1}](x-u) \right\} = 0. \quad (33)$$

In consequence, Eqs. (31)-(33) warrant continuity for the term $\partial[e^{ik_0 r} r^n]/\partial x$ for all $\hat{x} \neq \hat{y}$, and also for $\hat{x} = \hat{y}$, when considering this term as function of $\hat{x}$. Nevertheless, due to the radial symmetry of $e^{ik_0 r}[n r^{n-1} + ik_0 r^n]/r$ in Eq. (31), term $\partial[e^{ik_0 r} r^n]/\partial x$, interpreted as function of $\hat{y}$, must be also a continuous function. The same reasoning applies to $\partial[e^{ik_0 r} r^n]/\partial y$ and $\partial[e^{ik_0 r} r^n]/\partial z$; therefore, if $f_\gamma(r)$ is a polynomial of powers $n \geq 2$ for r values such that $0 \leq r < \gamma$, then $\nabla e^{ik_0 r} f_\gamma(r)$ will be continuous for all points $\hat{y}$ corresponding to those r values. Moreover, without loss of generality, we can assume that $p_\gamma(r) = ar^2 + br^3 + cr^4 + dr^5$ with $p_\gamma(\gamma/2) = p_0(\gamma/2)$, which implies that

$$p_\gamma(r) = (42/\gamma^3) r^2 - (111/\gamma^4) r^3$$
$$+ (102/\gamma^5) r^4 - (32/\gamma^6) r^5. \quad (34)$$

So, from Eqs. (30) and (34) we get

$$\nabla e^{ik_0 r} f_\gamma(r) = \begin{cases} e^{ik_0 r} A(r)(\hat{x} - \hat{y}) & r \geq \gamma \\ e^{ik_0 r} B(r)(\hat{x} - \hat{y}) & 0 \leq r < \gamma, \end{cases} \quad (35)$$

where $A(r) = [ik_0 r - 1]/r^3$ and

$$B(r) = \frac{84}{\gamma^3} - \frac{333r}{\gamma^4} + \frac{408r^2}{\gamma^5} - \frac{160r^3}{\gamma^6}$$
$$+ (ik_0)\left[\frac{42r}{\gamma^3} - \frac{111r^2}{\gamma^4} + \frac{102r^3}{\gamma^5} - \frac{32r^4}{\gamma^6}\right], \quad (36)$$

are two functions such that $\lim_{r\to\gamma^+} A(r) = A(\gamma) = B(\gamma) = \lim_{r\to\gamma^-} B(r)$, $\lim_{r\to\gamma^+} A'(r) = A'(\gamma) = B'(\gamma) = \lim_{r\to\gamma^-} B'(r)$, $\lim_{r\to 0^+} B(r) = B(0)$, and $\lim_{r\to 0^+} B'(r) = B'(0)$. Equation (35) is now a smooth function (C^1-class) for all $\hat{y} \neq \hat{x}$ and also for $\hat{y} = \hat{x}$. In the same way, term $e^{ik_0 r} f_\gamma(r)$ is another smooth function for all $\hat{y}$ by construction. From this construction and Eq. (25), we can calculate again

$$\int\limits_{\mathbb{R}^3}^{3} D(\hat{y} - \hat{x}) dV(\hat{y})$$
$$= -\frac{1}{4\pi} \lim_{\gamma\to 0^+} \left[\int\limits_{B_\varepsilon(\hat{x})}^{3} \nabla^2 e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \right.$$
$$\left. + k_0^2 \int\limits_{B_\varepsilon(\hat{x})}^{3} e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \right], \quad (37)$$

for any $\varepsilon > 0$ arbitrary and fixed. But this time we have

$$\int\limits_{B_\varepsilon(\hat{x})}^{3} e^{ik_0 r} f_\gamma(r) dV(\hat{y}) = \int\limits_{B_\varepsilon(\hat{x})\setminus B_\gamma(\hat{x})}^{3} \frac{e^{ik_0 r}}{r} dV(\hat{y})$$
$$+ \int\limits_{B_\gamma(\hat{x})}^{3} e^{ik_0 r} p_\gamma(r) dV(\hat{y})$$
$$= \frac{4\pi}{k_0^2} \left[e^{ik_0\varepsilon}(1 - ik_0\varepsilon) - e^{ik_0\gamma}(1 - ik_0\gamma) \right]$$
$$+ e^{ik_0 r_\bullet} p_\gamma(r_\bullet) \frac{4\pi\gamma^3}{3}, \quad (38)$$

by definition of f_γ when considering $\gamma < \varepsilon$. In the last expression, the integral of $(e^{ik_0 r}/r)$ can be calculated by a translation to the origin and spherical coordinates, while the integral of $e^{ik_0 r} p_\gamma(r)$ is obtained from the *mean value theorem for integrals* [21,36] due to the integrand's continuity. In this case $r_\bullet = ||\hat{x} - \hat{y}_\bullet||$ with $\hat{y}_\bullet \in B_\gamma(\hat{x}) \cup \partial B_\gamma(\hat{x})$, and the factor $(4\pi\gamma^3/3)$ is the volume of the ball $B_\gamma(\hat{x})$. On the other hand, when taking the complex module of the integral in Eq. (38), we have that

$$\left| \int\limits_{B_\varepsilon(\hat{x})}^{3} e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \right| \leq \frac{4\pi}{k_0^2} |e^{ik_0\varepsilon}(1 - ik_0\varepsilon)$$
$$- e^{ik_0\gamma}(1 - ik_0\gamma)| + \frac{4\pi\gamma^3}{3} |p_\gamma(r_\bullet)|. \quad (39)$$

Additionally, from Eq. (34), the triangle inequality, and the fact that $r_\bullet \leq \gamma$, it follows that

$$\frac{4\pi\gamma^3}{3} |p_\gamma(r_\bullet)| \leq \frac{1148\pi\gamma^2}{3}. \quad (40)$$

In an extreme case $(4\pi\gamma^3/3)|p_\gamma(r_\bullet)| \simeq (4\pi\gamma^2/3)$ for $r_\bullet$-values close or equal to γ, but the term $(4\pi\gamma^3/3)|p_\gamma(r_\bullet)|$ is always dominated by proportional factors to γ^2. Then, when considering very small values of γ (the limit context of Eq. (37)), the inequalities in Eqs. (39) and (40) permit us to establish that $e^{ik_0 r_\bullet} p_\gamma(r_\bullet)(4\pi\gamma^3/3) \approx 0$ and

$$k_0^2 \int\limits^{3}_{B_\varepsilon(\hat{x})} e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \approx 4\pi \left[e^{ik_0\varepsilon}(1 - ik_0\varepsilon) - 1 \right], \tag{41}$$

from Eq. (38). Now, due to the form of term $\nabla e^{ik_0 r} f_\gamma(r)$ in Eq. (35) and Proposition 2, it follows that $\nabla^2 e^{ik_0 r} f_\gamma(r) = \nabla \cdot \nabla e^{ik_0 r} f_\gamma(r) = -\nabla_{\hat{y}} \cdot \nabla e^{ik_0 r} f_\gamma(r)$. Therefore,

$$\begin{aligned} &\int\limits^{3}_{B_\varepsilon(\hat{x})} \nabla^2 e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \\ &= - \int\limits^{3}_{B_\varepsilon(\hat{x})} \nabla_{\hat{y}} \cdot \nabla e^{ik_0 r} f_\gamma(r) dV(\hat{y}) \\ &= - \int\limits^{2}_{\partial B_\varepsilon(\hat{x})} e^{ik_0 r} A(r)(\hat{x} - \hat{y}) \cdot \frac{(\hat{y} - \hat{x})}{||\hat{y} - \hat{x}||} dA(\hat{y}) \\ &= \cdots = 4\pi e^{ik_0\varepsilon}(ik_0\varepsilon - 1), \end{aligned} \tag{42}$$

as a consequence of the divergence Gauss theorem in variable $\hat{y}$, applied to the field $\nabla e^{ik_0 r} f_\gamma(r)$, and the formula for $\nabla e^{ik_0 r} f_\gamma(r)$ when $\gamma < \varepsilon = r$. Naturally, the reduction of the calculations in the double integral of Eq. (42) is implied from the fact that $\hat{y} \in \partial B_\varepsilon(\hat{x})$. So, since the smoothness of the field $\nabla e^{ik_0 r} f_\gamma(r)$ on $B_\varepsilon(\hat{x})$ is warranted in this case, then function G in Eq. (25) satisfies property 2), when considering Eqs. (41), (42), and (37). In conclusion, $G(\hat{x}, \hat{y})$ in Eq. (25), with f_γ in Eq. (30) and p_γ in Eq. (34), is a true Green function. And of course, this G is understood as the limit of a sequence of smooth functions that converge to $g_{\hat{y}}(\hat{x})$ almost everywhere. In mathematical terms, $G(\hat{x}, \hat{y}) = g_{\hat{y}}(\hat{x})$ for all $\hat{x} \neq \hat{y}$, except in $\hat{x} = \hat{y}$, where $g_{\hat{y}}(\hat{y})$ is not defined and $G(\hat{y}, \hat{y}) = 0$. Informally speaking, that is why $g_{\hat{y}}(\hat{x})$ inherits the name of its equivalent $G(\hat{x}, \hat{y})$.

Finally, we could be attempted to believe that G is an ordinary function like

$$G_\bullet(\hat{x}, \hat{y}) = \begin{cases} e^{ik_0||\hat{x}-\hat{y}||}/||\hat{x} - \hat{y}|| & \text{for} \quad \hat{x} \neq \hat{y} \\ 0 \quad \text{for} \quad \hat{x} = \hat{y}, \end{cases} \tag{43}$$

and that is true, in some way, with respect to the image sets induced by both expressions. Although formally

$$\int\limits^{3}_{\mathbb{R}^3} L[G_\bullet] dV(\hat{y}) = 0,$$

there is no obstacle to call $G_\bullet$ as a Green function because G is also a *generalization* of $G_\bullet$ or, equivalently, $G_\bullet = g_{\hat{y}}$ almost everywhere. Something similar happens when thinking in the one dimensional step function, also known as the Heaviside step. The derivative of the Heaviside step is zero at all points, except in the jump discontinuity where it is not defined. The integral of this derivative along the real line is identically zero. However, if the Heaviside step is understood as a distribution, then its *weak derivative* [26] is a Dirac delta (see the remarks in Appendix A). As well known, the integral of this weak derivative along the real line is identically one. These affirmations may seem contradictory, but they are only a question of abstract interpretation.

5. The integral theorem of Helmholtz and Kirchhoff and the retarded potential

Let us consider an elemental region Ω in $\mathbb{R}^3$ as a closed set, in such a way that its boundary is conformed by two independent surfaces. This means that $\partial\Omega = S \cup \partial B_\varepsilon(\hat{x}_0)$ with $S \cap \partial B_\varepsilon(\hat{x}_0) = \emptyset$, where S is a surface that bounds $\partial B_\varepsilon(\hat{x}_0)$ and $\emptyset$ denotes the empty set. Also, we are going to assume that S is smooth by parts, where $S \subset \Omega$ and $\partial B_\varepsilon(\hat{x}_0) \subset \Omega$, due to the fact that Ω is closed. So, region Ω can be assumed as a glass ovoid with an inside air sphere, where the elliptical surface of the ovoid corresponds to S, and the surface of the air sphere is $\partial B_\varepsilon(\hat{x}_0)$. It should be understood that $\partial\Omega$ limits two exterior zones and one inner zone (see Fig. 1): the big exterior zone given by the open set $(\Omega \cup B_\varepsilon(\hat{x}_0))^c = \{\hat{x} \in \mathbb{R}^3 : \hat{x} \notin \Omega \cup B_\varepsilon(\hat{x}_0)\}$, the small exterior zone given by $B_\varepsilon(\hat{x}_0)$, and the inner zone $\Omega \setminus \partial\Omega = \Omega \setminus (S \cup \partial B_\varepsilon(x_0)) = \{\hat{x} \in \Omega : \hat{x} \notin S \cup \partial B_\varepsilon(x_0)\}$.

Therefore, when considering possible parametrizations of the surfaces S and $\partial B_\varepsilon(\hat{x}_0)$, we must think that the unitary normals $\hat{n}(\hat{x})$ should point towards the big exterior zone when $\hat{x} \in S$, and point towards the small exterior zone when $\hat{x} \in \partial B_\varepsilon(\hat{x}_0)$, respectively. So, if we apply the second Green identity to a C^1-phasor $f(\hat{x}) = a(\hat{x}) e^{i\phi(\hat{x})}$, corresponding to some μ solution of the homogeneous wave equation on Ω, and the function $g(\hat{x}) = e^{ik_0 r}/r$ with $r = ||\hat{x} - \hat{x}_0||$, then we get

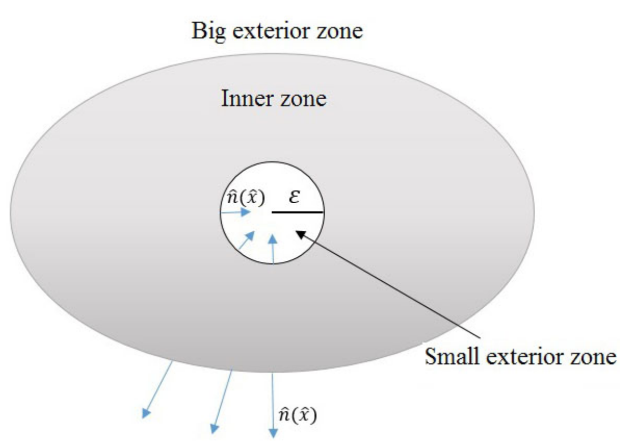

FIGURE 1. Sketch of the region Ω. In this example, it is assumed as an ovoid with an inside sphere of radius ε.

$$\int\limits_{\partial\Omega}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = \int\limits_{\Omega}^{3} (f\nabla^2 g - g\nabla^2 f) dV. \quad (44)$$

However,

$$\int\limits_{\Omega}^{3} (f\nabla^2 g - g\nabla^2 f) dV = \int\limits_{\Omega}^{3} (-k_0^2 fg + k_0^2 gf) dV = 0, \quad (45)$$

because both the phasor f and function g satisfy the Helmholtz equation for all points in Ω. This implies that

$$\int\limits_{\partial\Omega}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = \int\limits_{S}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$
$$+ \int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = 0, \quad (46)$$

or, equivalently,

$$\int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$
$$= -\int\limits_{S}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA. \quad (47)$$

Since the double integral on the right hand side of Eq.(47) is a constant, independently of the ε value, it follows that

$$\lim_{\varepsilon \to 0^+} \int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA =$$
$$-\int\limits_{S}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA, \quad (48)$$

including the possibility that $\partial\Omega = S \cup \{\hat{x}_0\}$ for the limit case. Now, the integral on $\partial B_\varepsilon(\hat{x}_0)$ in Eqs.(47) and (48) can be expressed as

$$\int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$
$$= 4\pi\varepsilon^2 \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right)\bigg|_{\hat{x}_*}, \quad (49)$$

from the mean value theorem of integrals, where $4\pi\varepsilon^2$ is the area of $\partial B_\varepsilon(\hat{x}_0)$ and $\hat{x}_*$ is some fixed point in $\partial B_\varepsilon(\hat{x}_0)$. Thus, if $\hat{x}_* \in \partial B_\varepsilon(\hat{x}_0)$, then

$$g(\hat{x}_*) = \frac{e^{ik_0\varepsilon}}{\varepsilon}, \quad (50a)$$

$$\nabla g(\hat{x}_*) = e^{ik_0\varepsilon} \left(\frac{ik_0\varepsilon - 1}{\varepsilon^3} \right) (\hat{x}_* - \hat{x}_0), \quad (50b)$$

by using the formulas in Proposition 1. Owing to the fact that vector $\hat{n}$ points towards the small exterior zone in this particular case, we have that

$$\frac{\partial g}{\partial n}(\hat{x}_*) = \nabla g \cdot \hat{n}|_{\hat{x}_*}, \quad (51a)$$

$$\hat{n}(\hat{x}_*) = \frac{\hat{x}_0 - \hat{x}_*}{||\hat{x}_0 - \hat{x}_*||} = -\frac{\hat{x}_* - \hat{x}_0}{\varepsilon}, \quad (51b)$$

then

$$\frac{\partial g}{\partial n}(\hat{x}_*) = \frac{e^{ik_0\varepsilon}(1 - ik_0\varepsilon)}{\varepsilon^2}, \quad (52)$$

from Eqs. (50) and (51). Hence

$$\int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA$$
$$= 4\pi \left[f(\hat{x}_*) e^{ik_0\varepsilon}(1 - ik_0\varepsilon) - \varepsilon e^{ik_0\varepsilon} \frac{\partial f}{\partial n}(\hat{x}_*) \right], \quad (53)$$

after substituting Eqs. (50) and (52) into Eq. (49). If $\varepsilon \to 0^+$, then $\hat{x}_* \to \hat{x}_0$ and

$$\lim_{\varepsilon \to 0^+} \int\limits_{\partial B_\varepsilon(\hat{x}_0)}^{2} \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n} \right) dA = 4\pi f(\hat{x}_0), \quad (54)$$

from Eq. (53) and the fact that f and $\partial f/\partial n$ are continuous functions (f is C^1-function). The limit in Eq. (54) represents the same to write

$$f(\hat{x}_0) = \frac{1}{4\pi} \int\limits_{S}^{2} \left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n} \right) dA, \quad (55)$$

by considering Eqs. (48) and (54). Equation (55) is valid for any C^1-phasor $f : \mathbb{R}^3 \to \mathbb{C}$, where $g(\hat{x}) = e^{ik_0 r}/r$ and $r = ||\hat{x} - \hat{x}_0||$. In this case $S \cup \{\hat{x}_0\} = \partial\Omega$ is the union of a smooth-by-parts surface S with the frontier point $\hat{x}_0$. Examples of smooth-by parts surfaces could be a sphere, an ellipsoid, the three faces of triangular pyramid, the six faces of a parallelepiped, etc. So, without loss of generality and in the context of *distributions*, the result in Eq. (55) is also valid when simply assuming $S = \partial\Omega$ in such a way that $\hat{x}_0 \in \Omega \backslash S$. Equation (55) corresponds to the *integral theorem of Helmholtz and Kirchhoff*, which can be generalized to any g function satisfying the Helmholtz equation in an equivalent sense. In other words, g in Eq. (55) could be any Green function. In this argumentation we have assumed that $L[f] = 0$ for all $\hat{x} \in \Omega$, but such hypothesis can be modified to an equivalent case as follows: Let G be a Green function with respect to the linear operator L, and let h be a distribution similar to $h_0\delta(\hat{x} - \hat{x}_1)$ with a constant factor (an independent term with respect to variable $\hat{x}$) h_0, and $\hat{x}_1 \notin \Omega \setminus \partial\Omega$. Thus, if f is a function such that $L[f(\hat{x})] = h(\hat{x})$, then

$$f(\hat{y}) = \frac{1}{4\pi} \overset{2}{\int\limits_{\partial\Omega}} \Big(G(\hat{x}, \hat{y}) \frac{\partial f}{\partial n}(\hat{x}) - f(\hat{x}) \frac{\partial G}{\partial n}(\hat{x}, \hat{y}) \Big) dA(\hat{x}), \tag{56}$$

for all $\hat{y} \in \Omega \setminus \partial\Omega$. Equation (56) can be simply deduced, for instance, from the second Green identity we have

$$\overset{2}{\int\limits_{\partial\Omega}} \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dA(\hat{x}) = \overset{3}{\int\limits_{\Omega}} (G\nabla^2 f - f\nabla^2 G) dV(\hat{x})$$
$$= \overset{3}{\int\limits_{\Omega}} G(\nabla^2 + k_0^2) f dV(\hat{x}) - \overset{3}{\int\limits_{\Omega}} f(\nabla^2 + k_0^2) G dV(\hat{x}), \tag{57}$$

but this is the same that

$$\ldots = -4\pi \overset{3}{\int\limits_{\Omega}} G L[f] dV(\hat{x}) + 4\pi \overset{3}{\int\limits_{\Omega}} f L[G] dV(\hat{x})$$
$$= -4\pi \overset{3}{\int\limits_{\Omega}} G(\hat{y}, \hat{x}) h(\hat{x}) dV(\hat{x}) + 4\pi \overset{3}{\int\limits_{\Omega}} f(\hat{x}) \delta(\hat{x} - \hat{y}) dV(\hat{x})$$
$$= -4\pi \overset{3}{\int\limits_{\Omega}} G(\hat{y}, \hat{x}) h(\hat{x}) dV(\hat{x}) + 4\pi f(\hat{y}), \tag{58}$$

from definition of G. Then, from Eqs.(57) and (58) we get

$$f(\hat{y}) = \frac{1}{4\pi} \overset{2}{\int\limits_{\partial\Omega}} \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dA(\hat{x}) + \overset{3}{\int\limits_{\Omega}} G(\hat{y}, \hat{x}) h(\hat{x}) dV(\hat{x}). \tag{59}$$

So, if $h(\hat{x}) = h_0 \delta(\hat{x} - \hat{x}_1)$ with $\hat{x}_1 \notin \Omega \setminus \partial\Omega$, then the triple integral on Ω in Eq. (59) is identically zero from the properties of the Dirac delta. This is because

$$\overset{3}{\int\limits_{\Omega}} G(\hat{y}, \hat{x}) \delta(\hat{x} - \hat{x}_1) dV(\hat{x}) = G(\hat{y}, \hat{x}_1),$$

only if $\hat{x}_1 \in \Omega \setminus \partial\Omega$. Moreover, for all $\hat{x} \in \Omega \setminus \partial\Omega$, the integrand $G(\hat{x}, \hat{y}) h(\hat{x}) = h_0 G(\hat{x}, \hat{y}) \delta(\hat{x} - \hat{x}_1) = 0$ does not have any discontinuity in $\Omega \setminus \partial\Omega$ when considering G in Eq. (25). Thus, by interchanging $\hat{x}$ by $\hat{y}$ and vice-versa in Eq. (59), we get a *potential solution* for the problem of solving $L[f(\hat{x})] = h(\hat{x})$ for all $\hat{x} \in \Omega \setminus \partial\Omega$. Such a problem would be solved with the next hypothesis: the knowledge of functions f and $\partial f / \partial n$ on $\partial\Omega$, a given function h, preferable smooth and defined on Ω, and a possible and convenient Green function G, defined with respect to the operator L. The inferred solution from Eq. (59) can be expressed as $f(\hat{x}) = f_0(\hat{x}) + f_1(\hat{x})$ with

$$f_0(\hat{x}) := \frac{1}{4\pi} \overset{2}{\int\limits_{\partial\Omega}} \left(G \frac{\partial f}{\partial n} - f \frac{\partial G}{\partial n} \right) dA(\hat{y}) \tag{60}$$

and

$$f_1(\hat{x}) := \overset{3}{\int\limits_{\Omega}} G(\hat{x}, \hat{y}) h(\hat{y}) dV(\hat{y}), \tag{61}$$

where

$$L\left[f_1(\hat{x})\right] = \overset{3}{\int\limits_{\Omega}} L[G(\hat{x}, \hat{y})] h(\hat{y}) dV(\hat{y})$$
$$= \overset{3}{\int\limits_{\Omega}} \delta(\hat{y} - \hat{x}) h(\hat{y}) dV(\hat{y}) = h(\hat{x}). \tag{62}$$

Since $L[f(\hat{x})] = h(\hat{x})$ by hypothesis, it follows that $L[f_0(\hat{x})] = 0$. In other words, f_0 and f_1 are particular solutions of the homogeneous and the non-homogeneous Helmholtz equation, respectively. We say that f_0 and f_1 are "particular" functions, because both of them depend on the G chosen. Eventually, beyond of considering a bounded set as Ω, if we think in solving $L[f] = h$ on $\mathbb{R}^3$, the result in Eq. (59) would be equivalent to consider

$$f(\hat{x}) = \overset{3}{\int\limits_{\mathbb{R}^3}} G(\hat{x}, \hat{y}) h(\hat{y}) dV(\hat{y}), \tag{63}$$

as the potential solution.

On the other hand, the theory of Green functions, exposed in a previous section, was described with respect to the operator $[-1/(4\pi)][\nabla^2 + k_0^2]$. However, this theory does not change if we consider other similar operators as $(\nabla^2 + k_0^2)$ or $[\nabla^2 + (k_0/c_0)^2]$. Thus, when considering $L := [-1/(4\pi)][\nabla^2 + (k_0/c_0)^2]$, function

$$g_0(\hat{x}) := \frac{-e^{i(-k_0/c_0)||\hat{x} - \hat{y}||}}{||\hat{x} - \hat{y}||}, \tag{64}$$

satisfies that

$$\left(\nabla^2 + \frac{k_0^2}{c_0^2} \right) g_0(\hat{x}) = 0, \tag{65}$$

when assuming $\hat{y}$ as a constant vector and $\hat{x} \neq \hat{y}$. To verify Eq. (65), it is only required to replace k_0 and $\hat{x}_0$ in Proposition 1 by $-k_0/c_0$ and $\hat{y}$, respectively. Now, we want to solve

$$\left(\frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mu(t, \hat{x}) = \zeta(t, \hat{x}), \tag{66}$$

on a domain Ω, where ζ (considered only as function of $\hat{x}$) behaves as a Dirac delta translated to some point outside of $\Omega \setminus \partial\Omega$. So, by proposing a solution of the form $\mu(t,\hat{x}) = f(\hat{x})e^{ik_0 t}$ and substituting this solution in Eq. (66), we get

$$e^{ik_0 t}\left(-\nabla^2 - \frac{k_0^2}{c_0^2}\right) f(\hat{x}) = \zeta(t,\hat{x}). \tag{67}$$

Therefore, when Eq.(67) is evaluated at $(t-(||\hat{x}-\hat{y}||/c_0),\hat{x})$, it is obtained that

$$e^{ik_0(t-(||\hat{x}-\hat{y}||/c_0))}\left(-\nabla^2 - \frac{k_0^2}{c_0^2}\right) f(\hat{x}) = \zeta(t-(||\hat{x}-\hat{y}||/c_0),\hat{x}). \tag{68}$$

In some way, the equality in Eq. (67) suggests that ζ could be interpreted as a function with separable variables, time t, and position $\hat{x}$. This is similar to conveniently think that $\zeta(t,\hat{x}) = P(t)Q(\hat{x})$, then Eq. (67) can be expressed as $L[f(\hat{x})] = h(\hat{x})$, where $h(\hat{x}) = h_0 Q(\hat{x})$ and $h_0 = h_0(t) = P(t)e^{-ik_0 t}/(4\pi)$ is a constant term with respect to variable $\hat{x}$. Consequently, if the spatial part of ζ is a function like $Q(\hat{x}) = \delta(\hat{x}-\hat{x}_1)$ with $\hat{x}_1 \notin \Omega\setminus\partial\Omega$, then the phasor f can be calculated by the integral theorem of Helmholtz and Kirchhoff, the second Green identity, and any convenient Green function G, as

$$f(\hat{y}) = \frac{1}{4\pi}\overset{2}{\int\limits_{\partial\Omega}}\left(G\frac{\partial f}{\partial n} - f\frac{\partial G}{\partial n}\right) dA(\hat{x}) = \frac{1}{4\pi}\overset{3}{\int\limits_{\Omega}}(G\nabla^2 f - f\nabla^2 G)dV(\hat{x}), \tag{69}$$

for all $\hat{y} \in \Omega \setminus \partial\Omega$. In addition, when taking $G = g_0$, it is concluded that

$$f(\hat{y}) = \frac{1}{4\pi}\overset{3}{\int\limits_{\Omega}}\frac{e^{-ik_0||\hat{x}-\hat{y}||/c_0}}{||\hat{x}-\hat{y}||} \times \left(-\nabla^2 - \frac{k_0^2}{c_0^2}\right) f(\hat{x})dV(\hat{x}), \tag{70}$$

from Eqs.(64), (65), and (69). Furthermore, Eq.(70) implies that

$$\mu(t,\hat{y}) = \frac{1}{4\pi}\overset{3}{\int\limits_{\Omega}}\frac{1}{||\hat{x}-\hat{y}||}\zeta\left(t-\frac{||\hat{x}-\hat{y}||}{c_0},\hat{x}\right) dV(\hat{x}), \tag{71}$$

from the form assumed for μ, and Eqs. (68) and (70). The expression in Eq. (71) is known as the *retarded potential* of ζ, as mentioned in [5]. Its name reveals that the signal μ is recovered from the source ζ with a delay in time. This potential represents a standard solution of the *d'Alembert equation* displayed in Eq. (66). Moreover, the formula in Eq. (71) is independent of the fact that ζ behaves as a Dirac delta. For instance, if we simply assume that ζ is a smooth function on Ω, then Eq. (68) can be expressed as

$$L[f(\hat{x})] = h(\hat{x}) := \mathcal{H}\left(t - \frac{||\hat{x}-\hat{y}||}{c_0},\hat{x}\right), \tag{72}$$

where $\mathcal{H}(t,\hat{x}) = (e^{-ik_0 t}/(4\pi))\zeta(t,\hat{x})$. From Eq. (61), by interchanging $\hat{x}$ by $\hat{y}$, and considering $G = -g_0$, it follows that

$$f(\hat{y}) = \overset{3}{\int\limits_{\Omega}}\frac{e^{-i(k_0/c_0)||\hat{x}-\hat{y}||}}{||\hat{x}-\hat{y}||}\mathcal{H}\left(t-\frac{||\hat{x}-\hat{y}||}{c_0},\hat{x}\right) dV(\hat{x}) = \frac{e^{-ik_0 t}}{4\pi}\overset{3}{\int\limits_{\Omega}}\frac{1}{||\hat{x}-\hat{y}||}\zeta\left(t-\frac{||\hat{x}-\hat{y}||}{c_0},\hat{x}\right) dV(\hat{x}), \tag{73}$$

is a potential solution of the equation $L[f] = h$ given in Eq. (72). Indeed, from the form assumed for μ, Eq. (73) induces again the formula given in Eq. (71). Finally, if we want to solve $L[f] = h$ on $\mathbb{R}^3$, we could use Eq.(63) and conclude that

$$\mu(t,\hat{y}) = \frac{1}{4\pi}\overset{3}{\int\limits_{\mathbb{R}^3}}\frac{1}{||\hat{x}-\hat{y}||}\zeta\left(t-\frac{||\hat{x}-\hat{y}||}{c_0},\hat{x}\right) dV(\hat{x}), \tag{74}$$

in analogy to the previous formulas that preserve the name of *retarded potential*. Nevertheless, Eq. (74) is also valid for the case when ζ is a Dirac delta δ, specially in the case of approximating this δ with a sequence of smooth functions like the Gaussians.

6. Discussion and Conclusions

We have derived the retarded potential of a non-homogeneous wave equation by considering certain subtle mathematical details. These details refer to the use of distributions, in our own and simplified interpretation of these generalized functions, and in what sense it is said that a given function is a Green function. According to our analysis, we obtained a distributional solution f for Eq. (72), which permits us to construct a solution μ for Eq. (66). When considering Eq. (72) in a bounded set Ω, the solution can take place by establishing boundary conditions in the frontier $\partial\Omega$, as exposed by Eq. (59). These conditions may be imposed in f, or in $\partial f/\partial n$, depending on the problem. For instance, in the simple case of $h = 0$ for all points in Ω, it follows that $f = f_0$ from Eq. (60). In this particular case, if we only know f on the frontier $\partial\Omega$ (Dirichlet problem), then a desirable election of G could be a Green function that vanishes on this boundary.

In a general perspective, the base idea of the potential solutions is manifested by Eqs. (59)-(61) and (63), where the election of an appropriate Green function is crucial to obtain specific results. For example, the formula given by Eq. (71) in the bounded case, or by Eq. (74) in the unbounded case, respectively. The retarded potentials allow us to build solutions of classic problems in electrodynamics, wave propagators, radar, among others. For instance, potential theory can be applied for modeling equations related with the emission and detection of a SAR signal [5, 6]. In a SAR configuration, the main equation that involves the recovered values of the signal and the scattering object density is a consequence of using Eqs. (71) or (74), in connection with the first Born approximation. Such an approximation reduces the ill-posed problem of recovering the scattering object to a simple convolution-filtering problem.

About the mathematical rigor found in some references, an explicit and formal explanation about Green functions, by using operator theory, is found in [1]. Although in this reference there is no mention on that $g_{\hat{y}}(\hat{x}) = e^{ik_0 r}/r$ is Green function with respect to the Helmholtz operator, an extensive analysis to derive Green functions from many different linear operators is given. Nevertheless, such analysis is out of the scope of these notes. On the other hand, it is interesting to notice that expression $g_{\hat{y}}(\hat{x})$ is declared as a Green function in many books (for example [2–4, 8]), without any formal proof of that fact. Whereas in some other books and for some other kind of Green functions, a proof is provided but with drawbacks. For instance, the argumentation in [21] when justifying $f(\hat{x},\hat{y}) = -1/(4\pi||\hat{x}-\hat{y}||)$ as a Green function with respect to a distributional Poisson equation. In that reference, the drawback is exactly the same that was exposed in Sec. 4 on the illegal use of the divergence Gauss theorem. Of course, we do not pretend to criticize that books because they are actually excellent references and we are far from exposing a formal proof. However, we expect at least to motivate the reader on the importance of considering sequences of approximating functions [28], to have a more clear idea about the handling of distributions. Such as made in Eq. (25), by considering these kind of sequences, an equivalence relation could be established between $G(\hat{x},\hat{y})$, defined in Eq. (25), and function $g_{\hat{y}}(\hat{x})$. Since $G(\hat{x},\hat{y}) = g_{\hat{y}}(\hat{x})$ is valid almost everywhere with respect to the variable $\hat{y} \in \mathbb{R}^3$ (or $\hat{x} \in \mathbb{R}^3$) [26, 27], there is no doubt now of calling $g_{\hat{y}}(\hat{x})$ as a Green function. In a similar manner, the argumentation in [21] when justifying $f(\hat{x},\hat{y})$ as a Green function could be improved when considering sequences of smooth approximating functions. Just as Jackson explains in his analysis on Poisson and Laplace equations in reference [3], we want to emphasize a very good footnote which refers to the volume integral of Eq. (1.36) in that reference: "The reader may complain that (1.36) has been obtained in an illegal fashion since $1/|\mathbf{x}-\mathbf{x}'|$ is not well-behaved inside the volume V. Rigor can be restored by using a limiting process..." Well, such *limiting process* has been exemplified in this work.

Appendix A

Remarks:

I) Based on [28], definition of *distribution* for the general case of functions $h : \mathbb{R}^3 \to \mathbb{C}$ is given as follows: Let $\mathcal{D}$ be the set of C^∞-functions ϕ such that, ϕ, with all its derivatives vanish at infinity as fast as $||\hat{x}||^{-N}$ when $||\hat{x}|| \to \infty$, and independently of how long is the positive integer N. Any function ϕ in the set $\mathcal{D}$ is said to be *particularly well-behaved* and such set would correspond to the space of *test functions* defined in [26]. According to [28], a sequence of functions $\{h_\gamma\}$ (considering values $\gamma > 0$) is said to be *regular*, if and only if $\lim_{\gamma\to 0^+} \int^3_{\mathbb{R}^3} h_\gamma \phi dV$ exists, for all $\phi \in \mathcal{D}$. Thus, a *distribution* H is a regular sequence of functions in $\mathcal{D}$ given by $\{h_\gamma\}$, where symbol

$$\int\limits_{\mathbb{R}^3}^{3} H\phi dV$$

means

$$\lim_{\gamma\to 0^+} \int\limits_{\mathbb{R}^3}^{3} h_\gamma \phi dV. \tag{A1}$$

Of course, this symbol is not a true integral, because H is a sequence. Even in the case when H is interpreted as the limit of this sequence, such limit could not be an ordinary function and that is why H is declared as a *symbolic function* [28]. In a more strict sense, a *distribution*, is a continuous linear functional T_H defined on the space of functions $\mathcal{D}$ [1, 26], as the symbolic notation

$$T_H(\phi) := \int\limits_{\mathbb{R}^3}^{3} H\phi dV$$

suggests.

II) In this discussion, we have relaxed the hypothesis by considering $\mathcal{D}$, as the set conformed by functions that are at least of C^1-class and that vanish at infinity in the next weak sense [26]: function h vanishes at infinity, if and only if $V[S_{t,h}]$ is finite, for all constant $t > 0$. Here, $S_{t,h} := \{\hat{x} : |h(\hat{x})| > t\}$ and $V[S_{t,h}]$ is the volume (or *Borel measure*) of the set $S_{t,h}$. Moreover, in this work, a *distribution* H is understood as the limit of a sequence $\{h_\gamma\}$, conformed by elements in the set $\mathcal{D}$, our particular set of *well-behaved* or *test* functions. However, for practical purposes, our test functions do not require to satisfy more properties, like the regularity condition imposed in [28]. Conceptually, we are assuming that

$$H(\hat{x}) := \lim_{\gamma\to 0^+} h_\gamma(\hat{x}). \tag{A2}$$

Therefore, when an ordinary function h is such that $h(\hat{x}) = H(\hat{x})$ almost everywhere, we say that h is the distribution H.

III) When considering $\{e^{ik_0 r} f_\gamma(r)\}$ with $r = ||\hat{x}||$ in Eq. (25) (the simple case when $\hat{y} = \hat{0}$), we clearly have that functions $h_\gamma(\hat{x}) := e^{ik_0 r} f_\gamma(r)$ are of C^1-class, as described in Sec. 4. Now, since $|h_\gamma(\hat{x})| = |f_\gamma(r)| = f_\gamma(r)$, then $S_{t,h_\gamma} = S_{t,\gamma} := \{\hat{x} : f_\gamma(r) > t\}$. Hence, given a pair of constants $t, \gamma > 0$, it is not difficult to observe that $V[S_{t,\gamma}]$ is finite in any case: a) For $2/\gamma \le t$, there is no $\hat{x} \in \mathbb{R}^3$ such that $f_\gamma(r) > t$ because the maximum value of f_γ is achieved at $r = \gamma/2$, then

$S_{t,\gamma} = \emptyset$ and $V[\emptyset] = 0$. b) For $1/\gamma \le t < 2/\gamma$, we have $S_{t,\gamma} = B_{r_2}(\hat{0}) \setminus \overline{B_{r_1}(\hat{0})}$ for some fixed values r_1 and r_2 such that $0 < r_1 < r_2 \le \gamma$. Here, $\overline{B_{r_1}(\hat{0})} = B_{r_1}(\hat{0}) \cup \partial B_{r_1}(\hat{0})$ and that implies $S_{t,\gamma} \subset B_{r_2}(\hat{0}) \subset B_\gamma(\hat{0})$, which means that $V[S_{t,\gamma}] \le V[B_{r_2}(\hat{0})] \le V[B_\gamma(\hat{0})] = (4\pi\gamma^3/3)$. c) For $t < 1/\gamma$, we have $S_{t,\gamma} = B_{1/t}(\hat{0}) \setminus \overline{B_{r_1}(\hat{0})} \subset B_{1/t}(\hat{0})$ for some fixed value r_1 such that $0 < r_1 < 1/t$, then $V[S_{t,\gamma}] \le V[B_{1/t}(\hat{0})] = (4\pi/3t^3)$.

Therefore, remark III) establishes that the sequence of functions used in Eq. (25) is a sequence of well-behaved functions, a sequence in the set $\mathcal{D}$ defined in remark II).

IV) According to our definition of $\mathcal{D}$, any function ϕ in this set has at least a continuous partial derivative $\partial\phi/\partial x$ (it could be also with respect to y, or z). Then, a sequence of C^2-class functions in $\mathcal{D}$ given by $\{h_\gamma\}$, could induce a sequence $\{\partial h_\gamma/\partial x\}$ contained in $\mathcal{D}$. If so, we can talk about $\partial_w H/\partial x = \lim_{\gamma\to 0^+} \partial h_\gamma/\partial x$ and $H = \lim_{\gamma\to 0^+} h_\gamma$, independently if these limits represent functions or not. Therefore, if the sequences $\{\partial h_\gamma/\partial x\}$ and $\{h_\gamma\}$ are such that

$$\int_{\mathbb{R}^3}^{3} (\partial_w H/\partial x)\phi dV = -\int_{\mathbb{R}^3}^{3} H(\partial\phi/\partial x) dV, \qquad \text{(A3)}$$

for all $\phi \in \mathcal{D}$ (by considering the symbolic representation in Eq. (A1)), then $\partial_w H/\partial x$ is said to be a *weak derivative* of H in a distributional sense. Of course, we have used the notation $\partial_w H/\partial x$, to distinguish this limit (or generalized function) from the usual partial derivative of H, which is $\partial H/\partial x$ every time that H is interpreted as an ordinary function ($H = h$).

Proposition 1 *The function $g(\hat{x}) = e^{ik_0 r}/r$, where $r = ||\hat{x} - \hat{x}_0|| > 0$ and $\hat{x}_0$ is a constant position, satisfies*

$$\nabla g(\hat{x}) = e^{ik_0 r}\left(\frac{ik_0 r - 1}{r^3}\right)(\hat{x} - \hat{x}_0). \qquad \text{(A4)}$$

Moreover, such function satisfies the Helmholtz equation $(\nabla^2 + k_0^2)g(\hat{x}) = 0$ for all $\hat{x} \ne \hat{x}_0$.

Proposition 2 *Let us consider the variables $\hat{x} = (x, y, z)$, $\hat{y} = (u, v, w)$, and the operator $\nabla_{\hat{y}} := (\partial/\partial u, \partial/\partial v, \partial/\partial w)$. Thus, for any vectorial field of the form $H(r)(\hat{x} - \hat{y})$, with $r = ||\hat{x} - \hat{y}||$ and $H(r)$ as a smooth function for all $r > 0$, we have that*

$$\nabla_{\hat{y}} \cdot H(r)(\hat{x} - \hat{y}) = -\nabla \cdot H(r)(\hat{x} - \hat{y}). \qquad \text{(A5)}$$

In particular, function $g_{\hat{y}}(\hat{x}) = e^{ik_0||\hat{x}-\hat{y}||}/||\hat{x} - \hat{y}||$ satisfies that $\nabla g_{\hat{y}}(\hat{x}) = e^{ik_0 r}[(ik_0 r - 1)/r^3](\hat{x} - \hat{y})$, as a direct consequence of Proposition 1 (when $\hat{x}_0 = \hat{y}$). Therefore,

$$\nabla_{\hat{y}} \cdot \nabla g_{\hat{y}}(\hat{x}) = -\nabla \cdot \nabla g_{\hat{y}}(\hat{x}), \qquad \text{(A6)}$$

for $\hat{x} \ne \hat{y}$.

The last two propositions can be demonstrated by a careful calculation of partial derivatives and an adequate use of the chain rule.

Acknowledgment

The authors thank Dr. Gerardo García-Almeida for their valuable comments on this paper.

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